2.4 Moment Generating Function

The *moment generating function* (*m.g.f.*) of a random variable X is defined as $M_X(t) = E(e^{tX})$.

The m.g.f. is said to exist (i.e., converges to a finite value) if there exists some real constant $c(>0)$ such that $M_X(t)$ is finite for $|t| < c$.

At $t = 0$, m.g.f. always exists and equals 1.

The m.g.f. of a random variable X is given by

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & \text{if } X \text{ is continuous,} \end{cases}$$

where $f(x)$ is the p.m.f. or p.d.f. of X, according as X is discrete or continuous.

Note : M.g.f. may not always exist for all values of t. However, when we make use of the m.g.f., we always assume that it exists.

The importance of studying m.g.f. lies in the fact that the moments of different orders of the distribution of X can be obtained by differentiating $M_X(t)$, successively.

The rth order raw moment $(r > 0)$ of a random variable X can be obtained by finding the rth derivative of $M_X(t)$ with respect to t, evaluated at $t = 0$ (vide Result 2.1).

Properties of moment generating function :

Result 2.1 $M_X^{(r)}(0) = E(X^r) = \mu_r'$

Proof : $M_X(t) = E(e^{tX})$

$$= E\left[1 + tX + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^r}{r!} + \cdots\right], \text{ using Maclaurin series expansion of } e^{tX}$$

$$= 1 + t \cdot E(X) + \frac{t^2}{2!} E(X^2) + \cdots + \frac{t^r \cdot E(X^r)}{r!} + \cdots$$

$$= 1 + t\,\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!} \cdot \mu_r' + \cdots, \text{ a function of } t.$$

Now, $M_X'(t) = \dfrac{dM_X(t)}{dt} = E(X) + tE(X^2) + \dfrac{t^2}{2!}E(X^3) + \cdots$

Setting $t = 0$,

$M_X'(0) = E(X) = \mu_1'$, the mean of X.

Similarly,

$M_X''(t) = \dfrac{d^2}{dt^2} M_X(t) = E(X^2) + t\,E(X^3) + \dfrac{t^2}{2!} E(X^4) + \cdots$

At $t = 0$,

$M_X''(0) = E(X^2) = \mu_2'$.

(Note that the variance of X, $V(X) = \mu_2' - \mu_1'^2 = M_X''(0) - (M_X'(0)^2)$

Proceeding in the same manner we have,

$$M_X^{(r)}(t) = \frac{d^r}{dt^r} M_X(t)$$
$$= E(X^r) + t\,E(X^{r+1}) + \frac{t^2}{2!} E(X^{r+2}) + \cdots$$

Putting $t = 0$ we get

$$M_X^{(r)}(0) = \frac{d^r}{dt^r} M_X(t)\Big|_{t=0} = E(X^r) = \mu'_r.$$

Note : The central moments can also be obtained from $M_{X-\mu}(t)$, where μ is the mean of the distribution, exactly in the similar manner, since $M_{X-\mu}(t)$ can be written as follows :

$$M_{X-\mu}(t) = E\left(e^{t(X-\mu)}\right)$$
$$= E\left[1 + \frac{(X-\mu)t}{1!} + \frac{(X-\mu)^2 t^2}{2!} + \cdots + \frac{(X-\mu)^r t^r}{r!} + \cdots\right]$$
$$= 1 + t \cdot E(X-\mu) + \frac{t^2}{2!} E(X-\mu)^2 + \cdots + \frac{t^r}{r!} E(X-\mu)^r + \cdots$$
$$= 1 + t \cdot \mu_1 + \frac{t^2}{2!}\mu^2 + \cdots + \frac{t^r}{r!} \cdot \mu_r + \cdots$$

Then

$$\frac{d^r}{dt^r} M_{X-\mu}(t)\Big|_{t=0} = E(X-\mu)^r = \mu_r \cdot$$

Now we illustrate the method of finding moments of random variable *X*.

Example 2.23 Let the p.m.f. $f(x)$ of a random variable *X* is given by

$$f(x) = \begin{cases} 0.2 & \text{when } x = -1 \\ 0.3 & \text{when } x = 0 \\ 0.5 & \text{when } x = 1 \end{cases}$$

Find the m.g.f. of *X*. Hence find first four raw moments of *X*. What is the variance of *X* ?

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} f(x)$$
$$= e^{-t} \cdot 0.2 + e^0 \cdot 0.3 + e^t \cdot 0.5$$

Then

$$\mu'_1 = \frac{dM_X(t)}{dt}\Big|_{t=0} = 0.5e^t - 0.2e^{-t}\Big|_{t=0} = 0.5 - 0.2 = 0.3$$

$$\mu_2' = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = 0.5e^t + 0.2e^{-t}\Big|_{t=0} = 0.7$$

$$\mu_3' = \left.\frac{d^3 M_X(t)}{dt^3}\right|_{t=0} = 0.5e^t - 0.2e^{-t}\Big|_{t=0} = 0.3$$

$$\mu_4' = \left.\frac{d^4}{dt^4} M_X(t)\right|_{t=0} = 0.5e^t + 0.2e^{-t}\Big|_{t=0} = 0.7$$

And the variance of X is

$$\mu_2 = \mu_2' - \mu_1'^2 = 0.7 - (0.3)^2 = 0.61.$$

Result 2.2. *Let the random Variable X have the m.g.f.* $M_X(t)$. *Let* $Y = \alpha + \beta X$. *Then the m.g.f. of the random variable Y is given by*

$$M_Y(t) = e^{\alpha t} \cdot M_X(\beta t)$$

α, β *being the real constants.*

Proof: $M_Y(t) = E\left(e^{(\alpha+\beta X)t}\right)$

$$= E\left(e^{\alpha t} \cdot e^{(\beta t)X}\right)$$
$$= e^{\alpha t} \cdot E\left(e^{(\beta t)X}\right)$$
$$= e^{\alpha t} \cdot M_X(\beta t).$$

Note : 1. In particular, $M_{\beta X}(t) = M_X(\beta t)$

2. If $Z = \dfrac{X-\mu}{\sigma}$, μ, σ being the mean and standard deviation of X, respectively, then $X = \mu + \sigma Z$, and hence by Result 2.2

$$M_X(t) = e^{\mu t} \cdot M_Z(\sigma t).$$

This result is used to determine m.g.f. of a normal variable X having parameters μ and σ^2, from the m.g.f. of a standard normal variable Z. (vide Result 3.27)

Result 2.3 *Suppose that X and Y are two independent random variables having m.g.f.* $M_X(t)$ *and* $M_Y(t)$, *respectively. Let* $Z = X + Y$. *If* $M_Z(t)$ *be the m.g.f. of Z, then*

$$M_Z(t) = M_X(t) \cdot M_Y(t)$$

Proof : $M_Z(t) = E(e^{tZ}) = E(e^{t(X+Y)}) = E(e^{tX} \cdot e^{tY})$

$= E(e^{tX}) \cdot E(e^{tY})$, since X and Y are independent.

$= M_X(t) \cdot M_Y(t)$

Note : 1. This result can be extended to n independent variables $X_1, X_2, \cdots, X_n$ with m.g.f's $M_{X_1}(t)$, $M_{X_2}(t), \cdots, M_{X_n}(t)$, respectively.

Let $Z = \sum_{i=1}^{n} X_i$. Then $M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t)$.

2. This result can be used to establish the *reproductivity property* for a number of important distributions (such as binomial, Poisson, normal).

This property states : *If two or more independent random variables having a certain distribution are added, the resulting random variable has a distribution of same type as that of the summands. (vide Results 3.7, 3.15, 3.33)*

Result 2.4 *Let X and Y be two random variables with m.g.f's* $M_X(t)$ *and* $M_Y(t)$, *respectively. If* $M_X(t) = M_Y(t)$ *for all values of t, then X and Y have the same probability distribution.*

Implication : M.g.f. uniquely determines the probability distribution of the corresponding random variable, provided m.g.f. exists, i.e., there exists a one to one correspondence between the m.g.f. and the distribution function of a random variable.